Solving a Stochastic Inverse Problems using an itearive method

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Abstract

In this paper, we consider a linear equation $Ax = u$, where A is a compact operator in Hilbert space H and its inverse A^{-1} is defined on the set $Im(A) \subsetneq H$. To solve this problem arising from many experimental fields of science, we propose an iterative method with Gaussian errors which converges almost completely.

Key words: Inverse problem; Linear operator; Tikhonov regularization; Iterative methode.

Introduction

To solve problems that are not well-posed and inverse problems, the most used regularization method is Tikhonov regularization. This is one of the most known regularization methods in statistics as well in numerical analysis. In different methods of Thikhonov regularization, we always use the inversion of the operator, which is not always easy to do, this operation may be very costly in practice in terms of computing. Other methods have been proposed, such as the Landweber iterative method and semi-iterative and $\vartheta-$methods. In this paper, we propose an iterative method for solving a stochastic inverse problems. We establish exponential inequalities for the probability of the distance between the approximated solution given by an iterative method and the exact one. These inequalities yield the almost complete convergence and the convergence rate of approximate solution.

A large variety of problems can be often regarded mathematically as :

$$Ax = u \tag{1}$$

where $A : \mathbb{H} \to \mathbb{H}$ is a compact operator. x is unknown solution in Hilbert space $\mathbb{H}$ and $u \in \mathbb{H}$.

1 Setting of the problem

We consider the following iterative process, for any first iteration solution x_0, the n-th ($n \geq 1$) iteration solution is defined by

$$x_n = x_{n-1} - \epsilon \left(\lambda x_{n-1} - A^* \left(u_e - A x_{n-1} \right) \right) + a_n \xi_n \tag{2}$$

where $(a_i)_{i \geq 1}$ a sequence of real positive numbers such that na_n converges to a constant when n tends to infinity and $(\xi_i)_{i \geq 1}$ is a sequence of iid Gaussian, zero mean satisfying $\mathbb{E}\|\xi_i\|^2 = \sigma^2 < +\infty$.

Theorem 1 *Let A be a linear compact operator and x_e be the unique exact solution of equation* (1) *with exact right hand side u_e. Furthermore, considering the iterative process*

$$x_n = x_{n-1} - \epsilon\left(\lambda x_{n-1} - A^*\left(u_e - Ax_{n-1}\right)\right) + a_n\xi_n$$

where $(\xi_i)_{i\geq 1}$ is a sequence of iid Gaussian random elements, with zero. Assume that $1 < \epsilon\lambda < 2$. Then, $\forall\varepsilon > 0$

$$\mathbb{P}\left(\|x_n - x_e\| > \varepsilon\right) \leq K\exp\left(-\frac{0.0067\varepsilon^2\left\|B^{1-n}\right\|}{\left(\sum_{i=1} a_i^2\right)\sigma^4\|B\|^2}\right) \tag{3}$$

where $B = I - \epsilon\left(\lambda I + A^*A\right)$. And

$$K = \mathbb{E}\left(\exp\left(\frac{0.1201}{\sigma^2\|B\|^2}\|\xi_1\|^2\right)\right) < \infty$$

Corollary 2 *The sequence (x_n) converges almost completely (a.co.) to the exact solution x_e of the equation* (1).

Corollary 3 *We have*

$$x_n - x_e = O\left(\sqrt{\frac{\log n}{n}}\right) \qquad a.co. \tag{4}$$

Corollary 4 *For a given significance threshold γ, it exists an integer n_γ for which*

$$\mathbb{P}\left\{\left\|x_{n_\gamma} - x_e\right\| \leq \varepsilon\right\} \geq 1 - \gamma, \tag{5}$$

We have conducted a simulation study to show that the estimator x_n given by the iterative method (2) is consistent.

References

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