A new algorithm for solving convex quadratic programs

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1 Introduction

Quadratic programming is a mathematical discipline which is used in several applications such as regression, production, portfolio selection, SVM problems and partial differential equations, etc.

In this paper, instead of using the standard direction of the adaptive method [3], we suggest a new descent direction, called hybrid direction. We define a quantity called the optimality estimate from which we derive sufficient and necessary conditions for the optimality of a given support feasible solution. On the base of this new direction and following the work [1], which is devoted for solving linear programs, we construct an algorithm for solving the convex quadratic programming problem.

The paper is organized as follows: Section 2 states the problem and reviews some definitions. In Section 3, we present the suggested hybrid direction algorithm. Finally, Section 4 concludes the paper and provides some perspectives.

2 Statement of the problem and definitions

A convex quadratic program with bounded variables can be presented in the following form:

$$\min F(x) = \frac{1}{2}x'Dx + c'x, \text{ s.t. } Ax = b,\ d^- \leq x \leq d^+, \tag{1}$$

where $D = (d_{ij}, 1 \leq i, j \leq n)$ is a nonnul square matrix of dimension n, which is symmetric and positive semidefinite; $c,\ d^-,\ d^+$ and x are n-vectors; b is an m-vector; A is an $m \times n$ matrix, with $rank(A) = m < n$. The symbol (') represents the transposition operation. Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ be two sets of indices. Let J_B and J_N be two subsets of J such that $J = J_B \cup J_N,\ J_B \cap J_N = \emptyset,\ |J_B| = |I| = m$. Let $A_B = A(I, J_B)$ and $A_N = A(I, J_N)$. We assume that $\det A_B = \det A(I, J_B) \neq 0$. For a feasible solution $x \in \mathcal{R}^n$, we compute the reduced costs vector E as follows: $E' = g' - u'A$, with $g = g(x) = Dx + c,\ u' = g'_B A_B^{-1},\ g_B = g(J_B)$. Let I_N be the identity matrix of dimension $n - m$. So we compute the two matrices:

$$Z = Z(J, J_N) = \begin{pmatrix} -A_B^{-1}A_N \\ I_N \end{pmatrix},\ M = M(J_N, J_N) = Z'DZ. \tag{2}$$

Let $J_S \subset J_N$ such that $\det M_S = \det M(J_S, J_S) \neq 0$, and set $J_{NN} = J_N \setminus J_S$. So J_B and J_S are called respectively a constraints support (CS) and an objective function support (OS). The set $J_P = \{J_B, J_S\}$ formed by a CS J_B and an OS J_S is called a support for the problem (1). The pair $\{x, J_P\}$ is called a support feasible solution (SFS). It is called consistent if $E(J_S) = 0$ and it is called nondegenerate if $d_j^- < x_j < d_j^+,\ j \in J_B$. In the following, we assume that a CS J_B, an OS J_S, and a feasible solution $x = x(J)$, with $E(J_S) = 0$, are available in advance. So $\{x, J_P\}$ is a consistent SFS.

3 Algorithm

- We compute the number: $\alpha = \|D\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |d_{ij}| > 0$.
- We define the following sets of indices:

$$\begin{aligned} &J_{NN}^+ = \{j \in J_{NN} : E_j > \alpha(x_j - d_j^-)\},\ J_{NN}^- = \{j \in J_{NN} : E_j < \alpha(x_j - d_j^+)\}, \\ &J_{NN}^P = \{j \in J_{NN} : \alpha(x_j - d_j^+) \leq E_j \leq \alpha(x_j - d_j^-)\},\ J_{NN}^{P0} = \{j \in J_{NN} : E_j = 0\}, \\ &J_{NN}^{P+} = \{j \in J_{NN} : 0 < E_j \leq \alpha(x_j - d_j^-)\},\ J_{NN}^{P-} = \{j \in J_{NN} : \alpha(x_j - d_j^+) \leq E_j < 0\}. \end{aligned} \tag{3}$$

• We compute the nonnegative quantity $\beta(x, J_p, \alpha)$, called the *optimality estimate*:

$$\beta(x, J_P, \alpha) = \sum_{j \in J_{NN}^+} E_j(x_j - d_j^-) + \sum_{j \in J_{NN}^-} E_j(x_j - d_j^+) + \frac{1}{\alpha} \sum_{j \in J_{NN}^P} E_j^2. \tag{4}$$

We can prove the following theorem:

Theorem 1. *(Sufficient and necessary condition for optimality.) Let $\{x, J_P\}$ be a consistent SFS of the problem (1). Then the condition $\beta(x, J_P, \alpha) = 0$ is sufficient, and in the case of nondegeneracy of the SFS $\{x, J_P\}$, also necessary for the optimality of the feasible solution x.*

• If $\beta(x, J_P, \alpha) = 0$, then $\{x, J_P\}$ is an optimal SFS. Else, we compute the descent feasible direction, called hybrid direction, as follows:

$$\begin{aligned} &l_j = d_j^- - x_j, \text{ if } j \in J_{NN}^+;\ l_j = d_j^+ - x_j, \text{ if } j \in J_{NN}^-;\ l_j = \tfrac{-E_j}{\alpha}, \text{ if } j \in J_{NN}^P; \\ &l(J_S) = -M_S^{-1} M(J_S, J_{NN}) l(J_{NN}) \text{ and } l(J_B) = -A_B^{-1}(A_S l_S + A_{NN} l_{NN}). \end{aligned} \tag{5}$$

• We compute the reduced costs direction: $\delta_N = M l_N$.
• We compute the steplength θ^0 along the direction l as follows:

$$\begin{aligned} &\theta^0 = \min\{1, \theta_{j_1}, \theta_{j_s}, \sigma_F\},\ \theta_{j_1} = \min_{j \in J_B} \theta_j,\ \theta_{j_s} = \min_{j \in J_S} \theta_j,\ \sigma_F = \sigma_{j_*} = \min_{j \in J_{NN}} \sigma_j; \\ &\theta_j = \tfrac{d_j^+ - x_j}{l_j}, \text{ if } l_j > 0;\ \theta_j = \tfrac{d_j^- - x_j}{l_j}, \text{ if } l_j < 0;\ \theta_j = \infty, \text{ if } l_j = 0; \\ &\sigma_j = -\tfrac{E_j}{\delta_j}, \text{ if } E_j \delta_j < 0;\ \sigma_j = \infty, \text{ otherwise.} \end{aligned} \tag{6}$$

• We compute the new solution and the new reduced costs vector:

$$\bar{x} = x + \theta^0 l \text{ and } \bar{E}_N = E_N + \theta^0 \delta_N. \tag{7}$$

• Let us define the sets of indices: $\bar{J}_{NN}^+$, $\bar{J}_{NN}^-$, $\bar{J}_{NN}^{P0}$, $\bar{J}_{NN}^{P+}$, $\bar{J}_{NN}^{P-}$, which are obtained by replacing in (3), x by $\bar{x}$ and E by $\bar{E}$. Moreover, we compute the new optimality estimate $\beta(\bar{x}, J_P, \alpha)$ by replacing in (4), x, E, J_{NN}^+, J_{NN}^-, J_{NN}^P by $\bar{x}$, $\bar{E}$, $\bar{J}_{NN}^+$, $\bar{J}_{NN}^-$, $\bar{J}_{NN}^P$ respectively.
• If $\beta(\bar{x}, J_P, \alpha) = 0$, then $\{\bar{x}, J_P\}$ is an optimal SFS. Else, we start a new iteration with the SFS $\{\bar{x}, \bar{J}_P\}$, where $\bar{J}_P$ is the new support computed as follows:
• if $\theta^0 = \theta_{j_s}$, then set $\bar{J}_B = J_B, \bar{J}_S = J_S \setminus j_s$, $\bar{J}_P = \{\bar{J}_B, \bar{J}_S\}$;
• if $\theta^0 = \theta_F = \sigma_{j_*}$, then set $\bar{J}_B = J_B, \bar{J}_S = J_S \cup j_*$, $\bar{J}_P = \{\bar{J}_B, \bar{J}_S\}$;
• if $\theta^0 = \theta_{j_1}$, then
  • compute the vector $X' = (x_{j_1 j}, j \in J_N) = -e_{j_1}' A_B^{-1} A_N$;
  • if there exists an index $j_0 \in J_S$ such that $x_{j_1 j_0} \neq 0$, then set

$$\bar{J}_B = (J_B \setminus j_1) \cup j_0,\ \bar{J}_S = (J_S \setminus j_0),\ \bar{J}_P = \{\bar{J}_B, \bar{J}_S\};$$

  • else, set $\bar{J}_B = (J_B \setminus j_1) \cup j_*$, $\bar{J}_S = J_S$, $\bar{J}_P = \{\bar{J}_B, \bar{J}_S\}$.

4 Conclusion

In this work, we have suggested a hybrid direction algorithm for solving convex quadratic programming problems with bounded variables. Necessary and sufficient conditions are derived to characterize the optimality of the current solution. In future work, we will implement and compare our method with the classical approaches.

References

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